Supersymmetric Non-local Gas Equation

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In this paper we study systematically the question of supersymmetrization of the non-local gas equation. We obtain both the $N = 1$ and the $N = 2$ supersymmetric generalizations of the system which are integrable. We show that both the systems are bi-Hamiltonian. While the $N = 1$ supersymmetrization allows the hierarchy of equations to be extended to negative orders (local equations), we argue that this is not the case for the $N = 2$ supersymmetrization. In the bosonic limit, however, the $N = 2$ system of equations lead to a new coupled integrable system of equations.

I. INTRODUCTION

The classical isentropic gas equations [1]

$$\begin{aligned} u_t + uu_x + \frac{1}{v}\, P_x &= 0, \\ v_t + (vu)_x &= 0, \end{aligned} \tag{1}$$

where u, v denote respectively the velocity and the density of the gas, are known to constitute an interesting class of dispersionless integrable systems when the pressure is a monomial function of the density. For example, $P = v^\gamma, \gamma \neq 0, 1$ corresponds to the polytropic gas, while $P = -\frac{1}{v}$ describes the Chaplygin gas [2, 3]. Both these systems are of hydrodynamic type [4], integrable and a lot is known about the properties of these systems. Recently, it was shown that the gas equation for the case $P = -\frac{1}{2}\left(\partial^{-1}v\right)^2$ (called the non-local gas equation) is also integrable and has a very rich algebraic structure [5, 6]. This system of equations arises in astrophysical models of dark matter [7]. In this paper, we will study the supersymmetrization of this system maintaining the integrability aspects of the model.

It is worth recalling that supersymmetrization of dispersionless systems is at best poorly understood at the present [8, 9]. For example, in the case of polytropic gas, an integrable supersymmetric hierarchy has been obtained only for $N = 1$ supersymmetrization besides the "trivial" susy-B supersymmetrization [10, 11] and the supersymmetrization of the Chaplygin gas [12] resembles a susy-B symmetrization [9]. Even in the case of the $N = 1$ supersymmetric polytropic gas (which is integrable), it is not known if it is a bi-Hamiltonian system. In contrast, we will show that integrable $N = 1$ and $N = 2$ supersymmetrizations are possible for the system of non-local gas dynamics. Furthermore, the $N = 1$ supersymmetric system possesses two Hamiltonian structures which are compatible so that it is truly a bi-Hamiltonian system and possesses all the rich algebraic structures of its bosonic counterpart. The $N = 2$ supersymmetrization, on the other hand, leads to a new system of coupled integrable equations in the bosonic limit.

The paper is organized as follows. In section **II**, we briefly recapitulate the essential features of the bosonic system and present results on the $N = 1$ supersymmetric generalization that is integrable. We obtain the bi-Hamiltonian structure, the Casimir functionals as well as the conserved charges (local and non-local) of the system. In section **III**, we present the essential results on the $N = 2$ supersymmetrization of the system preserving integrability. We obtain the bi-Hamiltonian structure and argue that the second structure has no Casimir functional so that the system of equations cannot be extended to the negative orders. We present a brief summary of our results in section **IV**.

II. $N = 1$ SUPERSMMETRIZATION

The non-local gas equation is described by the system of equations

$$\begin{aligned} u_t &= -uu_x + \left(\partial^{-1}v\right), \\ v_t &= -\left(uv\right)_x . \end{aligned} \tag{2}$$

This system of equations is known to be integrable and a lot of the algebraic properties for the system are well known. For example, it is known that the system of equations (2) is a bi-Hamiltonian system with the two compatible Hamiltonian structures described by

$$\mathcal{D}_1 = \begin{pmatrix} 0 & -\partial \\ -\partial & 0 \end{pmatrix}, \qquad \mathcal{D}_2 = \begin{pmatrix} \partial^{-1} & -u_x \\ u_x & -(v\partial + \partial v) \end{pmatrix}. \tag{3}$$

The conserved charges satisfying the recursion relation can be constructed recursively and these charges are in involution by construction making the system integrable. Furthermore, the Hamiltonian structure $\mathcal{D}_1$ has three Casimir functionals (conserved charges whose gradients are annihilated by the Hamiltonian structures)

$$\begin{aligned}
H_1 &= \int \mathrm{d}x\ v,\\
H_1^{(1)} &= -\int \mathrm{d}x\ u_x \to 0,\\
H_1^{(2)} &= \int \mathrm{d}x\ u,
\end{aligned}\tag{4}$$

while the second Hamiltonian structure has a single Casimir functional

$$H_{-1} = 2\int \mathrm{d}x\ \left(v - \frac{1}{2}u_x^2\right)^{\frac{1}{2}}.\tag{5}$$

The existence of the Casimir functionals allows the hierarchy of flows to be extended to the negative order using the recursion relation and these are local equations unlike (2). In addition to the charges that are recursively constructed, the model also possesses two series of conserved charges whose gradients are not related by the recursion operator (and, therefore, are not in involution with the infinitely many charges which are in involution)

$$G_n = \int \mathrm{d}x\ u_x \left(\partial^{-1}v\right)^n, \qquad \tilde{G}_n = \int \mathrm{d}x\ \left[u^2 v + \frac{2}{2+n}\left(\partial^{-1}v\right)^2\right]\left(\partial^{-1}v\right)^n.\tag{6}$$

However, a scalar Lax representation for the system is not known which leads to difficulties in applying the standard techniques of supersymmetrizing the system. Therefore, we construct directly the $N=1$ supersymmetric extension of the non-local gas equation which is integrable as follows.

Let us introduce the two fermionic superfields

$$U = \psi + \theta u, \quad V = \chi + \theta v,\tag{7}$$

where θ represents the Grassmann coordinate of the supermanifold and we assume the canonical dimensions

$$[x] = -1, \quad [t] = 2, \quad [U] = \frac{1}{2}, \quad [V] = \frac{7}{2}.\tag{8}$$

Writing out the most general equation of dimension $\left[\frac{5}{2}\right]$, the one that leads to the correct bosonic limit (and is integrable) is given by

$$U_t = -U_x\,(DU) + \left(D^{-2}V\right),\tag{9}$$

$$V_t = -\left((DU)V\right)_x,\tag{10}$$

where

$$D = \frac{\partial}{\partial\theta} + \theta\frac{\partial}{\partial x}, \quad D^2 = \partial_x,\tag{11}$$

represents the supercovariant derivative on the superspace. In components, the equations take the forms

$$\begin{aligned}
u_t &= -uu_x + (\partial^{-1}v),\\
v_t &= -\left(uv + \psi_x\chi\right)_x,\\
\psi_t &= -\psi_x u + (\partial^{-1}\chi),\\
\chi_t &= -\left(u\chi\right)_x.
\end{aligned}\tag{12}$$

Clearly, this represents a nontrivial $N=1$ supersymmetrization (and not a B-supersymmetrization) of (2). However, it is not obvious immediately if this is integrable.

We note that the system of equations (10) is a Hamiltonian system of equations. In fact, the Hamiltonian

$$H_3 = \frac{1}{2}\int \mathrm{d}Z\ \left(V(DU)^2 + \left(D^{-2}V\right)\left(D^{-1}V\right)\right),\tag{13}$$

where $\mathrm{d}Z = \mathrm{d}x\mathrm{d}\theta$, together with the Hamiltonian structure

$$\mathcal{D}_1 = \begin{pmatrix} 0 & -D \\ -D & 0 \end{pmatrix}, \tag{14}$$

leads to (10) as

$$\begin{pmatrix} U_t \\ V_t \end{pmatrix} = \mathcal{D}_1 \begin{pmatrix} \frac{\delta H_3}{\delta U} \\ \frac{\delta H_3}{\delta V} \end{pmatrix}. \tag{15}$$

Similarly, it can also be checked that (10) can also be written as Hamiltonian equations with the Hamiltonian

$$H_2 = \int \mathrm{d}Z\ U\,(DV), \tag{16}$$

and the Hamiltonian structure

$$\mathcal{D}_2 = \begin{pmatrix} D^{-3} & -\frac{1}{2}\left(U_x + D^{-1}(DU_x)\right) \\ \frac{1}{2}\left(U_x + (DU_x)D^{-1}\right) & -\left(D(DV) + (DV)D - \frac{3}{2}DVD\right) \end{pmatrix}, \tag{17}$$

as

$$\begin{pmatrix} U_t \\ V_t \end{pmatrix} = \mathcal{D}_2 \begin{pmatrix} \frac{\delta H_2}{\delta U} \\ \frac{\delta H_2}{\delta V} \end{pmatrix}. \tag{18}$$

The Hamiltonian structures (14) and (17) have the necessary symmetry properties and it can be checked through the method of prolongation [13] that they satisfy Jacobi identity as well. Therefore, both $\mathcal{D}_1$ as well as $\mathcal{D}_2$ define genuine Hamiltonian structures. In fact, it is obvious that $\mathcal{D}_1$ satisfies Jacobi identity trivially. It can also be checked that the change of variables

$$\tilde{U} = U_x, \quad \tilde{V} = V - \frac{1}{2}(DU_x)U_x = V - \frac{1}{2}(D\tilde{U})\tilde{U}, \tag{19}$$

diagonalizes the second Hamiltonian structure which coincides with the supersymmetric $SL(2) \otimes U(1)$ algebra and satisfies Jacobi identity. More, importantly, it can also be checked through the method of prolongation that an arbitrary linear combination

$$\mathcal{D} = \mathcal{D}_2 + \alpha\mathcal{D}_1, \tag{20}$$

also defines a Hamiltonian structure (satisfies Jacobi identity) so that we conclude that the system of equations (10) is truly a bi-Hamiltonian system. It follows now from Magri's theorem that the supersymmetric system of equations is integrable.

From the two Hamiltonian structures in (14) and (17), we can obtain the recursion operator associated with the system defined as

$$\begin{aligned} \mathcal{R} &= \mathcal{D}_2\mathcal{D}_1^{-1} && (21) \\ &= \begin{pmatrix} \frac{1}{2}\left(U_x + D^{-1}(DU_x)\right)D^{-1} & -D^{-4} \\ (D^2V)D^{-1} + \frac{1}{2}(DV) + \frac{3}{2}VD & -\frac{1}{2}\left(U_x + D^{-1}(DU_x\right)D^{-1} \end{pmatrix}. && (22) \end{aligned}$$

This helps us determine the conserved quantities associated with the system recursively and the first few take the forms

$$\begin{aligned} H_1 &= \int \mathrm{d}Z\ V, \\ H_2 &= \int \mathrm{d}Z\ U(DV), \\ H_3 &= \frac{1}{2}\int \mathrm{d}Z\left[(DU)^2V + \left(D^{-1}V\right)\left(D^{-2}V\right)\right], \\ H_4 &= \int \mathrm{d}Z\left[(DU)^3V - 3U\left(D^{-1}V\right)^2 - 6(DU)V\left(D^{-3}V\right)\right], \\ &\ \vdots \end{aligned} \tag{23}$$

These charges are all conserved and are in involution with one another by construction reflecting the integrability of the system.

It is worth noting that all the charges in the series (23) are bosonic. In addition, we have found a charge that is fermionic and is conserved under the flow. It has the form

$$\tilde{H} = \int \mathrm{d}Z\ UV\left(D^{-2}V\right). \tag{24}$$

Furthermore, much like the bosonic system, we have also found two series of bosonic conserved charges that are not related recursively

$$\begin{aligned} G_n &= \int \mathrm{d}Z\ U(DV)(D^{-1}V)^n, \\ \tilde{G}_n &= \int \mathrm{d}Z\ (\partial^{-1}V)(D^{-1}V)^n\left[(n+2)\left(D^{-1}V\right)^2 + (n+1)(n+3)U_xV(DU) - (n+3)(D^{-1}U)(DU)(DU_x)\right]. \end{aligned} \tag{25}$$

Much like the bosonic system, the $N = 1$ supersymmetric system also has Casimir functionals whose gradients are annihilated by the two Hamiltonian structures of the system. It is easy to check that the three conserved quantities

$$\begin{aligned} H_1 &= \int \mathrm{d}Z\ V, \\ H_1^{(1)} &= -\int \mathrm{d}Z\ U_x \rightarrow 0, \\ H_1^{(2)} &= \int \mathrm{d}Z\ U, \end{aligned} \tag{26}$$

are conserved and are Casimir functionals of the first Hamiltonian structure (14), namely,

$$\mathcal{D}_1 \begin{pmatrix} \frac{\delta H_1}{\delta U} \\ \frac{\delta H_1}{\delta V} \end{pmatrix} = \mathcal{D}_1 \begin{pmatrix} \frac{\delta H_1^{(1)}}{\delta U} \\ \frac{\delta H_1^{(1)}}{\delta V} \end{pmatrix} = \mathcal{D}_1 \begin{pmatrix} \frac{\delta H_1^{(2)}}{\delta U} \\ \frac{\delta H_1^{(2)}}{\delta V} \end{pmatrix} = 0. \tag{27}$$

This is very much like in the bosonic model. Furthermore, the second Hamiltonian structure (17) also has a Casimir functional

$$H_{-1} = \int \mathrm{d}Z\ \frac{V - \frac{1}{2}(DU_x)U_x}{\sqrt{(DV) - \frac{1}{2}(DU_x)^2 - \frac{1}{2}U_{xx}U_x}}, \tag{28}$$

which is conserved and satisfies

$$\mathcal{D}_2 \begin{pmatrix} \frac{\delta H_{-1}}{\delta U} \\ \frac{\delta H_{-1}}{\delta V} \end{pmatrix} = 0. \tag{29}$$

The existence of Casimir functionals suggests that the hierarchy of equations generated by $H_n, n > 0$ can be extended to the negative orders through the inverse of the recursion operator (22). Formally, the inverse can be defined as

$$\mathcal{R}^{-1} = \mathcal{D}_1\mathcal{D}_2^{-1}. \tag{30}$$

However, it is worth noting here that unlike in the bosonic case where a closed form expression for the inverse exists, here we have not found such a form. However, one can easily construct the conserved charges associated with the negative order of the hierarchy using the recursion relation

$$\mathcal{D}_1 \begin{pmatrix} \frac{\delta H_{-n}}{\delta U} \\ \frac{\delta H_{-n}}{\delta V} \end{pmatrix} = \mathcal{D}_2 \begin{pmatrix} \frac{\delta H_{-n-1}}{\delta U} \\ \frac{\delta H_{-n-1}}{\delta V} \end{pmatrix}. \tag{31}$$

For example, this leads to the first few conserved local charges of the forms

$$H_{-1} = -\int \mathrm{d}Z\ \frac{\tilde{V}}{\sqrt{(D\tilde{V})}},$$

$$
\begin{aligned}
H_{-2} &= \frac{1}{2}\int \mathrm{d}Z\left[\frac{U_{xx}}{\sqrt{(D\tilde{V})}} - \frac{(DU_{xx})\tilde{V}}{(D\tilde{V})^{\frac{3}{2}}}\right],\\
H_{-3} &= \frac{1}{24}\int \mathrm{d}Z\left[\frac{\left(12U_{xx}(DU_{xx}) + 10\tilde{V}_{xx}\right)}{(D\tilde{V})^{\frac{3}{2}}} - \frac{9\tilde{V}\left((D\tilde{V}_{xx}) + (DU_{xx})^2\right)}{(D\tilde{V})^{\frac{5}{2}}}\right],\\
H_{-4} &= \frac{3}{2}\int \mathrm{d}Z\left[\frac{U_{xx}(D\tilde{V}_{xx}) + U_{xx}(DU_{xx})^2 - \tilde{V}_x(DU_{xx})}{(D\tilde{V})^{\frac{3}{2}}}\right.\\
&\qquad - \frac{9U_{xxx}(D\tilde{V}_x) - 5U_{xx}(D\tilde{V}_{xx}) - 5U_{xx}(DU_{xx})^2 - 10\tilde{V}_{xx}(DU_{xx}) - 6V(DU_{xxxx})}{(D\tilde{V})^{\frac{5}{2}}}\\
&\qquad \left. + \frac{15\tilde{V}(D\tilde{V}_{xx})(DU_{xx}) + 5\tilde{V}(DU_{xx})^3}{(D\tilde{V})^{\frac{7}{2}}}\right],\\
&\vdots
\end{aligned}
\tag{32}
$$

where $\tilde{V}$ is defined in (19). These charges, of course, satisfy the recursion relation (and, therefore, are in involution) and reduce to the known charges in the bosonic limit, but it is interesting to note that in the bosonic limit, the last two terms in H_{-4} vanish.

Given the Hamiltonians in the negative hierarchy, we can obtain the equations through the known Hamiltonian structures. We simply note that the lowest order equation in the negative hierarchy has the form

$$
\begin{aligned}
U_{t_{-1}} &= \frac{1}{4}D\left[\frac{2}{(D\tilde{V})^{\frac{1}{2}}} - \frac{3\tilde{V}\tilde{V}_x}{(D\tilde{V})^{\frac{5}{2}}}\right],\\
V_{t_{-1}} &= \frac{1}{8}\left[\frac{-4U_{xxx}}{(D\tilde{V})^{\frac{1}{2}}} + 6\left(\frac{U_{xx}}{(D\tilde{V})^{\frac{1}{2}}}\right)_x + 2D\left(\frac{(DU_{xx})}{(D\tilde{V})^{\frac{1}{2}}}\right)\right.\\
&\qquad \left. + \frac{6\tilde{V}\tilde{V}_x U_{xxx}}{(D\tilde{V})^{\frac{5}{2}}} - 9\left(\frac{\tilde{V}\tilde{V}_x U_{xx}}{(D\tilde{V})^{\frac{5}{2}}}\right)_x - 3D\left(\frac{\tilde{V}\tilde{V}_x(DU_{xx})}{(D\tilde{V})^{\frac{5}{2}}}\right)\right].
\end{aligned}
\tag{33}
$$

III. $N = 2$ SUPERSYMMETRIZATION

The $N = 2$ supersymmetrization of the non-local gas equation can now be obtained in a simple manner. Let us define bosonic superfields $\overline{U}, \overline{V}$ in the $N = 2$ extended superspace as

$$
\overline{U} = U_1 + \theta_2 U, \quad \overline{V} = V_1 + \theta_2 V, \tag{34}
$$

where U, V are the $N = 1$ superfields defined in the last section while U_1, V_1 represent two new $N = 1$ bosonic superfields. In this extended superspace, we can define two covariant derivatives as

$$
\begin{aligned}
D_1 &= \frac{\partial}{\partial\theta_1} + \theta_1\frac{\partial}{\partial x},\\
D_2 &= \frac{\partial}{\partial\theta_2} + \theta_2\frac{\partial}{\partial x},\\
D_1^2 &= D_2^2 = \frac{\partial}{\partial x}, \quad D_1D_2 + D_2D_1 = 0.
\end{aligned}
\tag{35}
$$

With these, the $N = 2$ supersymmetric non-local gas equation which reduces to the $N = 1$ system (10) and is integrable takes the form

$$
\begin{aligned}
\overline{U}_t &= -\left(D_1D_2\overline{U}\right)\overline{U}_x + \left(\partial^{-1}\overline{V}\right),\\
\overline{V}_t &= -\left(\overline{V}\left(D_1D_2\overline{U}\right)\right)_x + D_1D_2\left[\overline{V}\,\overline{U}_x + \left(D_2\overline{U}_x\right)\left(D_1\overline{U}_x\right)\overline{U}_x - \left(D_1D_2\overline{U}_x\right)\overline{U}_x^2\right].
\end{aligned}
\tag{36}
$$

The system of equations (36) can be written as a Hamiltonian system with

$$
\begin{aligned}
\mathcal{D}_1 &= \begin{pmatrix} 0 & D_1 D_2 \partial^{-1} \\ D_1 D_2 \partial^{-1} & 0 \end{pmatrix}, \\
H &= \frac{1}{6} \int \mathrm{d}\overline{Z} \left[\left(D_1 D_2 \overline{U}_x\right) \overline{U}_x^3 - 3 \left(D_1 D_2 \partial^{-1} \overline{V}\right) \left(2 \left(D_1 D_2 \overline{U}\right) \overline{U}_x - \left(\partial^{-1} \overline{V}\right)\right)\right], \qquad (37)
\end{aligned}
$$

(where $\mathrm{d}\overline{Z} = \mathrm{d}x \mathrm{d}\theta_1 \mathrm{d}\theta_2$) so that

$$
\begin{pmatrix} \overline{U}_t \\ \overline{V}_t \end{pmatrix} = \mathcal{D}_1 \begin{pmatrix} \frac{\delta H}{\delta \overline{U}} \\ \frac{\delta H}{\delta \overline{V}} \end{pmatrix}. \qquad (38)
$$

The Hamiltonian structure clearly has the necessary anti-symmetry properties and trivially satisfies the Jacobi identity.

The system of equations (36) has a second Hamiltonian description as well. It can be checked that the equations can be written in the Hamiltonian form

$$
\begin{pmatrix} \overline{U}_t \\ \overline{V}_t \end{pmatrix} = \mathcal{D}_2 \begin{pmatrix} \frac{\delta \overline{H}}{\delta \overline{U}} \\ \frac{\delta \overline{H}}{\delta \overline{V}} \end{pmatrix}. \qquad (39)
$$

with the Hamiltonian

$$
\overline{H} = \int \mathrm{d}\overline{Z} \ \left(D_1 D_2 \overline{U}\right) \left[\overline{V} - \frac{1}{2} D_1 \left(\overline{U}_x (D_2 \overline{U}_x)\right)\right], \qquad (40)
$$

and the second Hamiltonian structure $\mathcal{D}_2$ with the elements

$$
\begin{aligned}
(\mathcal{D}_2)_{11} &= -\partial^{-1} D_1^{-1} D_2^{-1}, \\
(\mathcal{D}_2)_{12} &= \frac{1}{2}\left(-2\overline{U}_x + D_1^{-1}(D_1 \overline{U}_x) + D_2^{-1}(D_2 \overline{U}_x) + 2 D_1^{-1} D_2^{-1}(D_1 D_2 \overline{U}_x)\right), \\
(\mathcal{D}_2)_{21} &= \frac{1}{2}\left(2\overline{U}_x + (D_1 \overline{U}_x) D_1^{-1} + (D_2 \overline{U}_x) D_2^{-1} - 2 (D_1 D_2 \overline{U}_x) D_1^{-1} D_2^{-1}\right), \\
(\mathcal{D}_2)_{22} &= \frac{1}{2}\left(-2\partial \overline{V} - 2\overline{V}\partial + D_1 \overline{V} D_1 + D_2 \overline{V} D_2 - \partial \overline{U}_x^2 D_1 D_2 + \overline{U}_x^2 \partial D_1 D_2 + D_2 \overline{U}_x^2 \partial D_1 - D_1 \overline{U}_x^2 \partial D_2\right) \\
&\quad + (D_1 D_2 \overline{U}_x) \partial^{-1} D_1 D_2 \overline{U}_x D_1 D_2. \qquad (41)
\end{aligned}
$$

The second Hamiltonian structure is quite complicated and one can check Jacobi identity through the method of prolongation. However, it is much easier to check this through a change of variables

$$
\tilde{\overline{U}} = \overline{U}_x, \qquad \tilde{\overline{V}} = \overline{V} - (D_1 D_2 \tilde{\overline{U}}) \tilde{\overline{U}} - \frac{1}{2} (D_1 \tilde{\overline{U}})(D_2 \tilde{\overline{U}}), \qquad (42)
$$

the second Hamiltonian structure coincides with the $N = 2$ supersymmetric generalization of the $SL(2) \otimes U(1)$ algebra and thereby satisfies the Jacobi identity. Furthermore, the compatibility of the two structures $\mathcal{D}_1$ and $\mathcal{D}_2$ can also be checked in a straightforward manner through prolongation. Therefore, it follows that the system of $N = 2$ supersymmetric equations (36) are integrable. The infinite set of conserved charges in involution can be constructed using the recursion relation. However, their forms are extremely complicated and are not very enlightening. So, we do not list them here.

The bosonic limit of (36) leads to a new and interesting coupled equation that is integrable. Introducing the notation that u_0, u_1 represent the bosonic variables of the superfield $\overline{U}$ and v_0, v_1 represent the bosonic variables of the superfield $\overline{V}$, the equations can be written as

$$
\begin{aligned}
u_{0,t} &= -u_1 u_{1,x} + \left(\partial^{-1} v_0\right), \\
u_{1,t} &= u_{0,xx} u_{0,x} - u_{1,x} u_1 + \left(\partial^{-1} v_1\right), \\
v_{0,t} &= u_{0,xxx} u_{0,x}^2 + u_{0,xx}^2 u_{0,x} - u_{1,x}^2 u_{0,x} - v_{0,x} u_1 + v_1 u_{0,x}, \\
v_{1,t} &= \left(u_{1,xx} u_{0,x}^2 + 2 u_{1,x} u_{0,xx} u_{0,x} - v_{0,x} u_{0,x} - v_1 u_1\right)_x. \qquad (43)
\end{aligned}
$$

This is a new integrable system of equations and has the interesting feature that the second Hamiltonian structure for this system does not have any Casimir functional. As a result, the second Hamiltonian structure for the $N = 2$ supersymmetric system in (41) does not also possess any Casimir functional (although the first structure does) and the system of equations (36) cannot be extended to the negative hierarchy.

IV. SUMMARY

In this paper we have systematically studied the supersymmetrization of the non-local gas equation preserving integrability. We obtain the $N = 1$ supersymmetric system of equations and show that it has two compatible Hamiltonian structures making it a bi-Hamiltonian system. We construct the conserved charges of the system and show that the two Hamiltonian structures possess Casimir functionals. As a result, the system of supersymmetric equations can be extended to negative orders and these give rise to local equations of motion. We also construct the $N = 2$ supersymmetric generalization of the non-local gas equation and show that it is a bi-Hamiltonian system. In the bosonic limit, this equation leads to a new coupled integrable system of equations. Furthermore, we argue that in the $N = 2$ supersymmetric case, while the first Hamiltonian structure possesses Casimir functionals, the second does not. As a result, these equations cannot be extended to negative orders.

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